TRIG. SUBSTITUTION (CONTINUED!)

HWK # 1 $\displaystyle\int \frac{dx}{\sqrt{19-x^2}}$

$\sqrt{a^2-x^2}$ use $x = a\sin\theta$

- $x = \sqrt{19}\sin\theta$ since $a^2 = 19$
- $dx = \sqrt{19}\cos\theta\, d\theta$
- $\sqrt{19-x^2} = \sqrt{19-(\sqrt{19}\sin\theta)^2}$

$$= \sqrt{19-19\sin^2\theta}$$

$$= \sqrt{19(1-\sin^2\theta)} = \sqrt{19}\sqrt{1-\sin^2\theta}$$

$$= \sqrt{19}\sqrt{\cos^2\theta} = \sqrt{19}\cos\theta$$

So, we have $\displaystyle\int \frac{dx}{\sqrt{19-x^2}}dx = \int \frac{\sqrt{19}\cos\theta\, d\theta}{\sqrt{19}\cos\theta} = \int 1\, d\theta$

$$= \theta + C$$

Since $x = \sqrt{19}\sin\theta \longrightarrow \frac{x}{\sqrt{19}} = \sin\theta \longrightarrow \boxed{\arcsin\left(\frac{x}{\sqrt{19}}\right) = \theta + C}$

Thus, $\boxed{\arcsin\left(\frac{x\sqrt{19}}{19}\right) + C}$

In general,

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\left(\frac{x}{a}\right) + C$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\left(\frac{x}{a}\right) + C$$

Remember that these expressions are obtained through the trig u-substitution!

#8 $\displaystyle\int \frac{x^2}{\sqrt{36-x^2}}\,dx$ ∘∘○ ○ $\left\{ \begin{array}{l} x = a\sin\theta \\ dx = a\cos\theta\, d\theta \end{array} \right)$

$x = 6\sin\theta \qquad dx = 6\cos\theta\, d\theta$

$\sqrt{36-x^2} = 6\sqrt{1-\sin^2\theta} = 6\cos\theta$

$$\Longrightarrow \int \frac{36\sin^2\theta}{6\cos\theta}\left(6\cos\theta\, d\theta\right)$$

$$\Longrightarrow \int 36\sin^2\theta\, d\theta$$

$\sin^2 x = \dfrac{1-\cos 2x}{2}$

Lowering power formula

$$\Longrightarrow 36\int \frac{1-\cos 2\theta}{2}\,d\theta = \frac{36}{2}\int (1-\cos 2\theta)\,d\theta$$

$$\Longrightarrow 18\left[\int d\theta - \int \cos 2\theta\, d\theta\right]$$

$\int \cos 2\theta\, d\theta$

$u = 2\theta \quad du = 2d\theta$

$\frac{1}{2}du = d\theta$

$= \int \cos u \left(\frac{1}{2}du\right)$

$= \frac{1}{2}\int \cos u\, du$

$= \frac{1}{2}(+\sin u) + C$

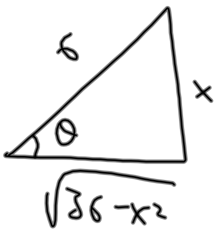
$$\Longrightarrow 18\left[\theta - \frac{1}{2}\sin 2\theta\right] + C$$

Recall $x = 6\sin\theta \longrightarrow \dfrac{x}{6} = \sin\theta$

$\longrightarrow \theta = \sin^{-1}\left(\dfrac{x}{6}\right)$

$$\Longrightarrow 18\left[\theta - \sin\theta\cos\theta\right] + C$$

$\sin 2\theta = 2\sin\theta\cos\theta$
(double angle formula)



$\cos\theta = \dfrac{\sqrt{36-x^2}}{6}$ and $\sin\theta = \dfrac{x}{6}$, $\theta = \sin^{-1}\left(\dfrac{x}{6}\right)$

Therefore, $18(\theta - \sin\theta\cos\theta) + C$

$$\boxed{= 18\left[\sin^{-1}\left(\frac{x}{6}\right) - \frac{x\sqrt{36-x^2}}{36}\right] + C}$$

8.5 Partial Fractions decomposition

Goal: $\int \frac{N(x)}{D(x)}\,dx$ $N(x), D(x)$ are polynomials:

$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$

Case 1:

If $\deg(N(x)) \geq \deg(D(x))$, $\frac{N(x)}{D(x)}$ is said to be proper $\Longrightarrow$ technique: long division!

Eg: $\int \frac{x^2+1}{x-1}\,dx$ "proper" because

$\deg(x^2+1) = 2 > 1 = \deg(x+1)$

We divide x^2+1 by $x-1$:

$$\begin{array}{r|l} & x+1 \\ \hline x-1 & x^2+0x+1 \\ & -(x^2-x) \\ \hline & \quad 1x+1 \\ & \quad -(1x-1) \\ \hline & \qquad 2 \end{array}$$

$$\frac{P(x)}{D(x)} = \frac{R(x)}{D(x)} + Q(x)$$

Q: quotient
D(x): divisor
R: remainder
P(x) = Polynomial

So $\frac{x^2+1}{x-1} = \frac{2}{x-1} + x + 1$

$$\Longrightarrow \int \frac{x^2+1}{x-1}\,dx = \int \frac{2}{x-1}\,dx + \int (x+1)\,dx$$

$$= 2\ln|x-1| + \frac{1}{2}x^2 + x + C$$

Case 2: "Improper" case $\Longrightarrow$ partial fractions!

For instance, if observed that:

$$\frac{-1}{x^2-1} = \frac{\frac{1}{2}?}{x+1} - \frac{\frac{1}{2}?}{x-1}$$

Then $\displaystyle\int \frac{-1}{x^2-1}\,dx = \frac{1}{2}\int \frac{1}{x+1}\,dx - \frac{1}{2}\int \frac{1}{x-1}\,dx$

$\displaystyle = \frac{1}{2}\ln|x+1| - \frac{1}{2}\ln|x-1| + C$

General forms for partial fractions

(1)
$$\frac{N(x)}{(Ax+B)^m} = \frac{A_1}{Ax+B} + \frac{A_2}{(Ax+B)^2} + \cdots + \frac{A_m}{(Ax+B)^m}$$

Sum of m-partial fractions for each repeated linear factors

(2)
$$\frac{N(x)}{(Ax^2+Bx+C)^n} = \frac{A_1X+B_1}{Ax^2+Bx+C} + \frac{A_2X+B_2}{(Ax^2+Bx+C)^2} + \cdots + \frac{A_nX+B_n}{(Ax^2+Bx+C)^n}$$

where AX^2+BxC is quadratic irreducible, ie, it is simple and it cannot be factored out without using complex numbers

Ex: Evaluate $\int \frac{2x+1}{x^2-7x+12}\,dx$

$$\frac{2x+1}{x^2-7x+12} = \frac{2x+1}{\underbrace{(x-3)(x-4)}_{\text{factored linear form}}} = \frac{A}{x-3} + \frac{B}{x-4}$$

find A, B

$$\Longrightarrow \frac{2x+1}{(x-3)(x-4)} = \frac{A(x-4)}{(x-3)(x-4)} + \frac{B(x-3)}{(x-4)(x-3)}$$

disregard the denominator $\Longrightarrow$

Top: $2x+1 = A(x-4) + B(x-3)$

- Let $x=3$ (b/c set $x-3=0$)

$$2(3)+1 = A(3-4) + B(3-3)$$

$$7 = A(-1) + 0 \longrightarrow \boxed{A=-7}$$

- Let $x=4$

$$9 = (0)A + B(1) \longrightarrow \boxed{B=9}$$

Therefore, $\int \frac{2x+1}{x^2-7x+12}\,dx = \int \frac{A}{x-3}\,dx + \int \frac{B}{x-4}\,dx$

$$= \int \frac{-7}{x-3}\,dx + \int \frac{9}{x-4}\,dx$$